Approximating a set of points by circles

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Abstract

This paper is an abstract of the German diploma thesis "Approximation von Punktmengen durch Kreise" finished by the author in March 2005. The thesis consists of two parts. The first part consists of some mathematical foundations about sets of points, circles and Voronoi diagrams in an Euclidian plane. Based on the characteristics of these geometric structures we determine the center and the radius of particular circles respectively following annalus: the largest empty circle, the smallest enclosing circle, the minimum-width annalus and the best fitting circle. The second part presents algorithms for solving these geometric problems (uniquely identifying the above geometrc objects for any finite set of points), implemented as Java applet.

1 Basics

We consider a set of points in the Euclidean plane and the nearest-point and furthest-point Voronoi diagram of bisectors.

Following properties of the nearest-point Voronoi diagram are useful for examining the position and radius of empty circles. Let S be a set of points in $\mathbb{R}^2$ and $\mathcal{VD}(\mathcal{S})$ the nearest-point Voronoi-Diagram. Let $x \in \mathbb{R}^2, r \in \mathbb{R}$ and $C(x, r)$ be a circle with center x and growing radius r starting with 0.

1. If the edge of $C(x, r)$ hits first one point of S, then $x \notin \mathcal{VD}(\mathcal{S})$.

2. If the edge of $C(x, r)$ hits first two points of S, then x lays on a Voronoi edge of $\mathcal{VD}(\mathcal{S})$.

3. If the edge of $C(x, r)$ hits first three or more points of S, then x lays on a Voronoi node of $\mathcal{VD}(\mathcal{S})$.

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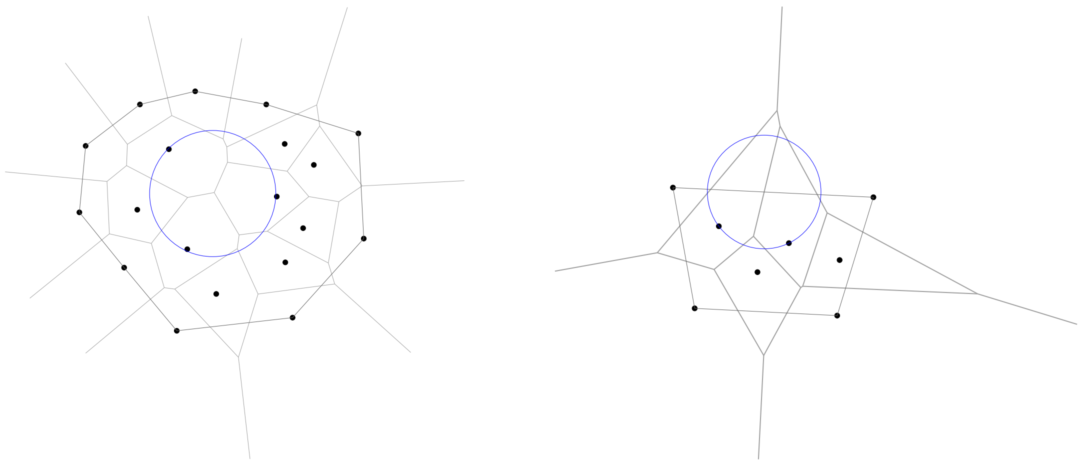
Analogous exist relations between the furthest-point Voronoi diagram and the position and radius of enclosing circles. Let S be a set of points in $\mathbb{R}^2$ and $\mathcal{FVD}(\mathcal{S})$ the furthest-point Voronoi-Diagram. Let $x \in \mathbb{R}^2, r \in \mathbb{R}$ and $C(x,r)$ be a circle with center x and shrinking radius r starting at ∞.

1. If the edge of $C(x,r)$ hits first one point of S, then $x \notin \mathcal{FVD}(\mathcal{S})$.

2. If the edge of $C(x,r)$ hits first two points of S, then x lays on a Voronoi edge of $\mathcal{FVD}(\mathcal{S})$.

3. If the edge of $C(x,r)$ hits first three or more points of S, then x lays on a Voronoi node of $\mathcal{FVD}(\mathcal{S})$.

If there's only the condition to maximize the radius of an empty circle to a set of points, there are an infinity number of circles which solve this problem and their radius is ∞. We define the largest empty circle based on two conditions. The center is inside the convex hull of the corresponding set of points and the radius has to be maximized. For this case the center lays on the nearest-point Voronoi diagram. Either the center is a Voronoi node or an intersection between the convex hull and a Voronoi edge.



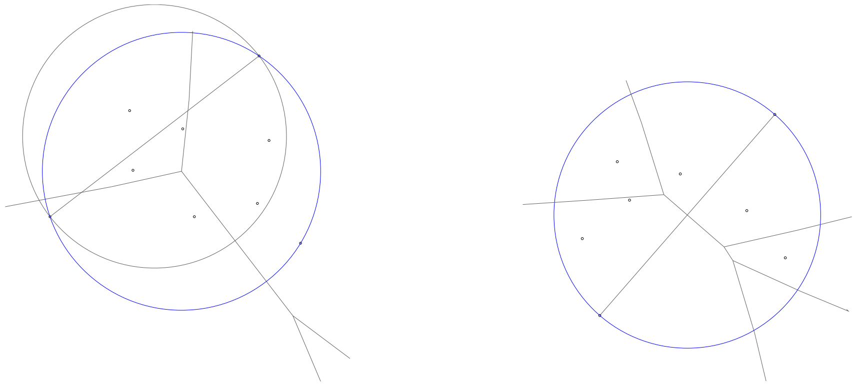
(a) center on a Voronoi node

(b) center on a Voronoi edge

Figure 1: Largest empty circle

The smallest enclosing circle to a set of points is unique. Its center is an element of the furthest-point Voronoi diagram. If a circle with radius equivalent to the diameter of the given set of points S encloses all points of S, this is the smallest enclosing circle. In this case, the center can be on a Voronoi edge or on a Voronoi node. If no circle with this radius encloses all points of S, then the center of the smallest enclosing circle lays on a Voronoi node.

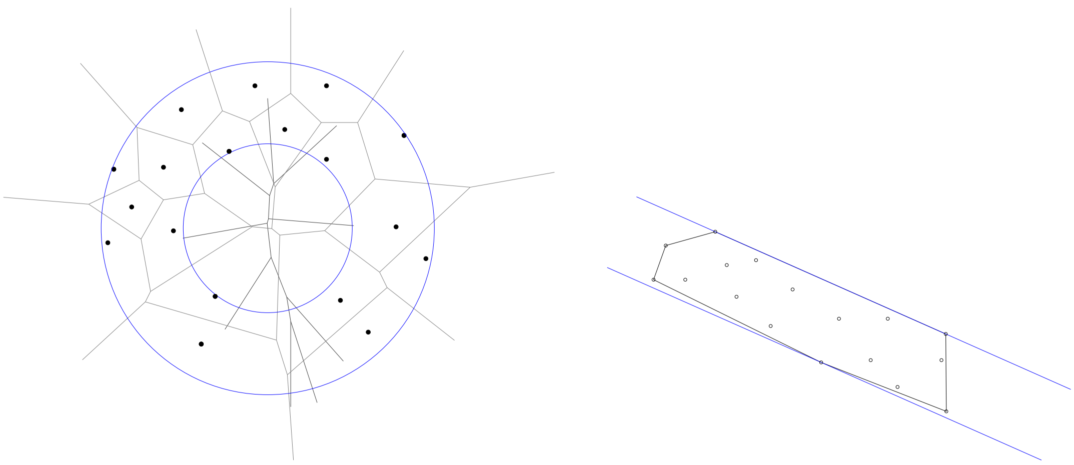
Two concentric circles define an annulus. If one of them is an empty and one of them is an enclosing circle, the annalus is additionally enclosing the given set of



(a) center on a Voronoi node (b) center on a Voronoi edge

Figure 2: Smallest enclosing circle

points. A minimum-width annalus contains all points and its width is minimized for any annalus existing under that condition. The center of such an annulus lays on an intersection between the nearest-point and furthest-point Voronoi diagram. If the width of a set of points is smaller than the width of any enclosing annalus, the minimum-width annalus is figured by a slab. A slab can be interpreted like an annalus with its center at ∞.



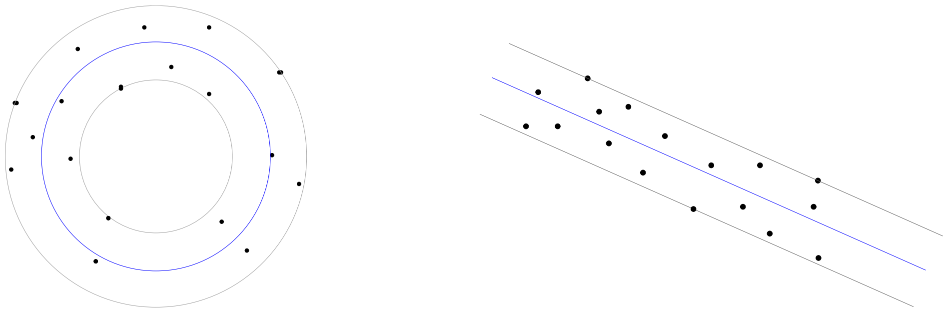
(a) center on an intersection of Voronoi diagrams (b) center at ∞

Figure 3: Minimum-width annalus

A best fitting circle can be considered under different conditions. We define it as the circle with minimized maximum radius.[1] If the minimum-width annalus is given, the circle has the same center and the radius amounts to the half of the

[1]In literature the expression best fitting circle is also found for a circle determined by least squares fitting.

inner radius added to the outer radius of the annalus. In the case that the width of the given set of points is smaller than the width of any enclosing annalus, the line in the middle of a slab fits the set the best.



(a) center on an intersection of Voronoi diagrams

(b) center at ∞

Figure 4: Best fitting circle

2 Algorithms

Corresponding to the defined circles, we have to construct first the convex hull, the nearest-point and the furthest-point Voronoi diagram to a given set of points. The convex hull is a double connected list and assists to calculate the diameter and the width of the given set of points among other things. We decided to use incremental algorithms for the Voronoi diagrams and the dual structures - the Delaunay triangulations - as data structures. The algorithms that construct the convex hull and the Voronoi diagrams can be done in time $O(n \log n)$ for a set of n points. The calculation of the diameter and the width runs in linear time.

A largest empty circle is found by examining the Voronoi nodes inside the convex hull and all intersections between the nearest-point Voronoi diagram and the convex hull. The circle with the largest radius has to be drawn. With the prerequisite that the nearest-point Delaunay triangulation and the convex hull is given, this algorithm can be carried out in time $O(n)$.

For the smallest enclosing circle, one diameter circle has to be checked if all points are included. If so, the smallest enclosing circle is found. If not so, all Voronoi nodes of the furthest-point Voronoi diagram have to be examined and the circle with the smallest radius has to be figured. This algorithm is done in linear time if the furthest-point Delaunay triangulation is known.

To determine the minimum-width annalus, the sweep line algorithm for finding all intersections of segments is adjusted to the problem to find all intersections of the Voronoi diagrams. The rays has to be treated differently than in the original algorithm. For all rays oriented into the direction ∞ corresponding to the x-axes, only the events for starting a segment are considered. All rays oriented into the direction $-\infty$ corresponding to the x-axes are artifically adjusted to segments. The starting point is found by examining all such rays with their direct neighboring rays whether or not there's an intersection. The smallest x-coordinate of an intersection is the start-coordinate for the sweep line. If there's no intersection, the sweep starts with lowest x-coordinate of an endpoint of the Voronoi edges. If the width of the found annalus is bigger than the width of the set of points, the slab has to be drawn otherwise the annalus. In worst-case there can be n^2 intersections between the Voronoi diagrams. Therefor this algorithm runs in time $O(n^2 \log n)$.

For figuring the best fitting circle is to check whether the minimum-width annalus is an annulus or a slab. Corresponding to the result a circle or a line has to be drawn.

The algorithms are mostly incremental because they are used for a interactive implementation. Besides we have to handle degenerate cases like collinear or cocircular sets of points in the implementation. We solved this problem by examining the sets of points and if need, treat them explicitely.